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L_p Markov–Bernstein Inequalities for Erdős Weights

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Let $W := e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, sufficiently smooth, and of faster than polynomial growth at infinity. We establish L_p Markov–Bernstein inequalities for Erdős weights; for example,

$$\|P'W\|_{L_p(\mathbb{R})} \leqslant CQ'(a_n)\|PW\|_{L_p(\mathbb{R})}$$

and

$$\left\|(P'W)(x)\left|1-\left(\frac{x}{a_n}\right)^2\right|^{1/2}\right\|_{L_p(\mathbb{R})} \leqslant C\frac{n}{a_n}\|PW\|_{L_p(\mathbb{R})},$$

for all polynomials P of degree at most n and $p \in (0, \infty)$. Here a_n is the Mhaskar–Rahmanov–Saff number for W. More general inequalities with L_p norms replaced by integrals of convex functions are established, as well as estimates of L_p Christoffel functions.

1. INTRODUCTION

In recent years, the subject of weighted approximation associated with weights W on $\mathbb{R}$ has received considerable attention [1, 13]. An essential ingredient of this theory is Markov–Bernstein inequalities, which relate the size of $P'W$ to the size of PW, for polynomials P. Typically, the weights considered have been *Freud weights*, that is, $W := e^{-Q}$, where Q is even, and of polynomial growth at infinity. The archetypal Freud weights are $W(x) := \exp(-|x|^\alpha)$, $\alpha > 0$.

The case where Q is of faster than polynomial growth at infinity was first treated by Erdős in a related context [2], and so for such Q, $W = e^{-Q}$ is called an *Erdős* weight. The approximation theory for Erdős weights has received relatively little attention, primarily because the necessary estimates (involving Christoffel functions and Markov–Bernstein inequalities) were lacking.

Recent progress has partly filled in this gap [3, 4, 7, 11]. It is the aim

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of this paper to exploit this to prove Markov–Bernstein inequalities and Christoffel function estimates in L_p and more general spaces, with a view to ultimately establishing Jackson–Bernstein approximation theorems.

Let $W := e^{-Q}$ be a sufficiently smooth Erdős weight, and for $u > 0$, let a_u be the uth Mhaskar–Rahmanov–Saff number for W, namely the root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)(1-t^2)^{-1/2} \, dt. \tag{1.1}$$

The significance of a_u is the Mhaskar–Saff identity

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n, a_n]}, \qquad \text{for all polynomials } P \text{ of degree } \leqslant n. \tag{1.2}$$

We shall show that for $0 < p < \infty$, and all polynomials of degree at most n,

$$\|P'W\|_{L_p(\mathbb{R})} \leqslant CQ'(a_n) \|PW\|_{L_p(\mathbb{R})}$$

and

$$\left\| (P'W)(x) \left| 1 - \left(\frac{x}{a_n}\right)^2 \right|^{1/2} \right\|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \|PW\|_{L_p(\mathbb{R})}.$$

Here C is independent of n and P, and (in contrast to the Freud case), $Q'(a_n)/(n/a_n)$ increases to infinity (but more slowly than any power of n) as $n \to \infty$.

In particular, the results apply to weights such as

$$W(x) := \exp(-\exp_k(|x|^\alpha)),$$

where $k \geqslant 1$, $\alpha > 1$, and $\exp_k$ denotes the kth iterated exponential. To those familiar with Freud weights, it is worth noting that Erdős weights are similar to weights on a finite interval in that they display "endpoint effects" that complicate matters.

Our results are stated in Section 2. Section 3 contains preliminaries and a proof of the L_p Christoffel function estimates. Section 4 contains the proof of the Bernstein inequalities.

2. Statement of Results

Throughout $\mathscr{P}_n$ denotes the class of real polynomials of degree at most n. Furthermore, $C, C_1, C_2, \ldots,$ denote positive constants independent of n, $P \in \mathscr{P}_n$, and $x \in \mathbb{R}$. The same symbol C or C_j does not necessarily indicate the same constant in different occurrences. We use $\sim$ as in [12]: We write $c_n \sim d_n$ if for some $C_1, C_2 > 0$,

$$C_1 \leqslant c_n/d_n \leqslant C_2, \qquad n \text{ large enough.}$$

Similarly we can define $f(x) \sim g(x)$.

Following is a suitable class of Erdős weights:

DEFINITION 2.1. Let $W := e^{-Q}$, where Q is even, continuous in $\mathbb{R}$, Q''' exists in $(0, \infty)$, and $Q'(x) > 0$, $x \in (0, \infty)$. Let

$$T(x) := 1 + xQ''(x)/Q'(x) = \frac{d}{dx}(xQ'(x))/Q'(x) \tag{2.1}$$

be increasing in $(0, \infty)$ with

$$\lim_{x \to 0+} T(x) > 1, \tag{2.2}$$

$$\lim_{x \to \infty} T(x) = \infty, \tag{2.3}$$

and for each $\varepsilon > 0$,

$$T(x) = O(Q'(x)^{\varepsilon}), \qquad x \to \infty. \tag{2.4}$$

Assume further that

$$Q''(x)/Q'(x) \sim Q'(x)/Q(x), \qquad x \text{ large enough,} \tag{2.5}$$

and for some $C > 0$,

$$|Q'''(x)|/Q'(x) \leqslant C\{Q'(x)/Q(x)\}^2, \qquad x \text{ large enough.} \tag{2.6}$$

Then we say that W is an *Erdős weight of class* 3, and write $W \in SE^*(3)$.

Remarks. (a) Some of the results do not require of W all of the above.

(b) It is (2.3) that forces Q to grow faster than any polynomial and so W to be an Erdős weight in the usual sense. By contrast for the Freud weight $W(x) = \exp(-|x|^{\alpha})$, $T(x) \equiv \alpha$.

(c) The condition (2.4) is a rather weak regularity condition, for one typically has for each $\varepsilon > 0$,

$$T(x) = O(\{\log Q'(x)\}^{1+\varepsilon}), \qquad x \to \infty.$$

(d) The class $SE^*(3)$ coincides with that in [11] and is a subclass of $SE(3)$ of [3]. It contains the most important Erdős weights

$$W(x) := \exp(-\exp_k(|x|^\alpha)),$$

$k \geqslant 1$, $\alpha > 1$, where

$$\exp_k(x) := \exp(\exp(\cdots \exp(x))) \qquad (k \text{ times}).$$

With extra effort, one can drop (2.2) and so also allow $\alpha > 0$. Another example of a weight in $SE^*(3)$ is

$$W(x) := \exp(-\exp\{\log(A + x^2)\}^\alpha),$$

$\alpha > 1$, A large enough.

An important special case of our results is:

THEOREM 2.2. *Let $W \in SE^*(3)$. For $n \geqslant 1$, let a_n be the positive root of* (1.1), *and let*

$$\varphi_n(x) := |1 - x^2| + T(a_n)^{-1}, \qquad x \in \mathbb{R}. \tag{2.7}$$

Let $0 < p < \infty$ and $\beta > 0$. Then for $n \geqslant 1$ and $P \in \mathscr{P}_n$,

$$\left\| (P'W)(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2} \right\|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \| PW \|_{L_p(\mathbb{R})}. \tag{2.8}$$

In particular,

$$\| P'W \|_{L_p(\mathbb{R})} \leqslant CQ'(a_n) \| PW \|_{L_p(\mathbb{R})} \tag{2.9}$$

and

$$\left\| (P'W)(x) \left| 1 - \left(\frac{x}{a_{\beta n}}\right)^2 \right|^{1/2} \right\|_{L_p(\mathbb{R})} \leqslant C \frac{n}{a_n} \| PW \|_{L_p(\mathbb{R})}. \tag{2.10}$$

We remark that

$$Q'(a_n) \sim \frac{n}{a_n} T(a_n)^{1/2}, \qquad n \geqslant 1, \tag{2.11}$$

so for "most x," (2.10) is superior to (2.9). One may think of (2.10) as an L_p Bernstein inequality and of (2.9) as an L_p Markov inequality: Their classical cousins on $[-1, 1]$ are respectively [8]

$$\| P'(x)(1 - x^2)^{1/2} \|_{L_p[-1,1]} \leqslant Cn \| P \|_{L_p[-1,1]}, \qquad P \in \mathscr{P}_n,$$

and

$$\|P'\|_{L_p[-1,1]} \leqslant Cn^2 \|P\|_{L_p[-1,1]}, \qquad P \in \mathscr{P}_n.$$

In this connection, it is instructive to note that the function $\varphi_n(x/a_{\beta n})^{1/2}$ plays much the same role for Erdős weights, as does

$$h_n(x) := (1-x^2)^{1/2} + 1/n, \qquad x \in [-1, 1] \tag{2.12}$$

for weights on $[-1, 1]$. By contrast, for Freud weights, there is no need for such a factor, as $T(a_n) \sim 1$, and $Q'(a_n) \sim n/a_n$.

The L_∞ analogue of (2.9) was obtained in [3, Theorem 2.6], [4, Theorem 1.3] and shown to be sharp in the sense that

$$\sup_{P \in \mathscr{P}_n} \|P'W\|_{L_\infty(\mathbb{R})}/\|PW\|_{L_\infty(\mathbb{R})} \sim Q'(a_n) \sim \frac{n}{a_n} T(a_n)^{1/2}.$$

Nikolskii inequalities and (2.10) for $p = \infty$ were obtained in [11, Theorem 1.5]. We believe that Theorem 2.2 is sharp with respect to the rate of growth of n.

We deduce Theorem 2.2 from

THEOREM 2.3. *Let $W \in SE^*(3)$ and $\{\varphi_n\}_{n=0}^{\infty}$ be as in* (2.7). *Let $\psi : [0, \infty) \to [0, \infty)$ be continuous, convex, non-negative, and non-decreasing with $\psi(0+) = \psi(0) = 0$. Let $0 < p < \infty$ and $\beta > 0$. Then for $n \geqslant 1$ and $P \in \mathscr{P}_n$,*

$$\int_{-\infty}^{\infty} \psi\left(\left\{|P'W|(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right\}^p\right) dx \leqslant C_1 \int_{-\infty}^{\infty} \psi\left(\left\{C_2 \frac{n}{a_n} |PW|(x)\right\}^p\right) dx. \tag{2.13}$$

A crucial role in our proofs is played by the following L_p–Christoffel function estimate:

THEOREM 2.4. *Assume the hypotheses of Theorem* 2.3. *Fix $l \geqslant 1$. Then for $P \in \mathscr{P}_{ln}$ and $x \in \mathbb{R}$,*

$$\psi\left(|PW|(x)^p\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right) \leqslant C_1 \frac{n}{a_n} \int_{-\infty}^{\infty} \psi(C_2\, |PW|(t)^p)\, dt. \tag{2.14}$$

Our method of proof is similar to that used in [6] or [8] for weights on $[-1, 1]$. We remark that Theorems 2.2 and 2.3 remain valid if for some fixed $l \geqslant 1$, we allow $P \in \mathscr{P}_{ln}$.

3. Proof of Theorem 2.4

Throughout the sequel, we assume that $W=e^{-Q}\in SE^*(3)$ and that $a_u=a_u(Q)$ is the root of (1.1) for $u>0$. Furthermore ψ: $[0,\infty)\to[0,\infty)$ denotes a continuous, convex, non-negative, and non-decreasing function with $\psi(0+)=\psi(0)=0$. We shall need several lemmas, the first listing elementary estimates for a_n, $T(a_n)$, etc.:

Lemma 3.1. (i) *Given* $\varepsilon>0$,

$$a_n=O(n^{\varepsilon}) \quad \textit{and} \quad T(a_n)=O(n^{\varepsilon}), \qquad n\to\infty. \tag{3.1}$$

(ii) *Given distinct* $\alpha,\beta>0$, *we have*

$$T(a_{\alpha n})\sim T(a_{\beta n}), \qquad n\to\infty, \tag{3.2}$$

$$\lim_{n\to\infty} a_{\alpha n}/a_{\beta n}=1, \tag{3.3}$$

and

$$|1-a_{\alpha n}/a_{\beta n}|\sim T(a_n)^{-1}, \qquad n\to\infty. \tag{3.4}$$

(iii) *For* $n\geqslant 1$,

$$Q'(a_n)\sim\frac{n}{a_n}T(a_n)^{1/2}. \tag{3.5}$$

(iv) *Given fixed* $k,l\geqslant 1$, *and* $\alpha,\beta>0$, *we have uniformly for* $x\in\mathbb{R}$ *and* $n\geqslant 1$,

$$\varphi_{kn}\left(\frac{x}{a_{\alpha n}}\right)\sim\varphi_{ln}\left(\frac{x}{a_{\beta n}}\right). \tag{3.6}$$

Proof. (i) First, $a_n=O(n^{\varepsilon})$ is (3.19) in [3, p. 19] or (2.20) in [4, p. 201]. The relation

$$T(a_n)=O(n^{\varepsilon})$$

follows from (2.25) in [4, p. 203] (note that $\chi\equiv T$ there).

(ii) First, (3.2) follows from (3.44) in [3, p. 23]. Second, (3.3) follows from (3.18) in [3, p. 19]. Next, (3.4) is (2.8) in [7, p. 260].

(iii) This is (3.15) in [3, p. 18] for $j=1$.

(iv) Recall first the definition (2.7) of φ_n. Then uniformly for $x \in \mathbb{R}$ and $n \geqslant 1$,

$$\varphi_{kn}\left(\frac{x}{a_{\alpha n}}\right) = \left|1 - \left(\frac{x}{a_{\alpha n}}\right)^2\right| + T(a_{kn})^{-1}$$

$$= \left|1 - \left(\frac{x}{a_{\beta n}}\right)^2 + \left(\frac{x}{a_{\beta n}}\right)^2 \left\{1 - \left(\frac{a_{\beta n}}{a_{\alpha n}}\right)^2\right\}\right| + T(a_{kn})^{-1}$$

$$\leqslant \left|1 - \left(\frac{x}{a_{\beta n}}\right)^2\right| + C\left|\frac{x}{a_{\beta n}}\right|^2 T(a_{ln})^{-1} + CT(a_{ln})^{-1}$$

(by (3.4) and (3.2))

$$\leqslant C_1 \left\{\left|1 - \left(\frac{x}{a_{\beta n}}\right)^2\right| + T(a_{ln})^{-1}\right\} + C\left[\left|\left(\frac{x}{a_{\beta n}}\right)^2 - 1\right| + 1\right] T(a_{ln})^{-1}$$

$$\leqslant C_2 \left\{\left|1 - \left(\frac{x}{a_{\beta n}}\right)^2\right| + T(a_{ln})^{-1}\right\} = C_2 \varphi_{ln}\left(\frac{x}{a_{\beta n}}\right). \quad \blacksquare$$

Next, we need an L_p infinite–finite range inequality:

LEMMA 3.2. *Let $0 < p \leqslant \infty$, and let*

$$\Delta_n := \left(\frac{\log n}{nT(a_n)}\right)^{2/3}, \qquad n \geqslant 1. \tag{3.7}$$

Then there exists $C > 0$, such that for $n \geqslant 1$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbb{R})} \leqslant (1 + n^{-2}) \|PW\|_{L_p[-a_n(1 + C\Delta_n),\, a_n(1 + C\Delta_n)]}. \tag{3.8}$$

In particular, given $r > 1$, we have for $n \geqslant n_1$ and $P \in \mathcal{P}_n$,

$$\|PW\|_{L_p(\mathbb{R})} \leqslant (1 + n^{-2}) \|PW\|_{L_p[-a_{rn},\, a_{rn}]}. \tag{3.9}$$

Proof. The inequality (3.8) is a special case of Theorem 5.2 in [3, p. 46]. Then (3.9) follows from the fact that (see (3.4))

$$a_{rn}/a_n - 1 \geqslant C_1/T(a_n) \geqslant C\Delta_n,$$

n large enough, by (3.1). $\blacksquare$

We shall use the above to prove an infinite–finite range inequality for integrals involving a convex function ψ instead of just pth powers. First, we need:

LEMMA 3.3 (Nikolskii Inequality). *Let* $0<p<\infty$. *For* $n\geqslant 1$ *and* $P\in\mathscr{P}_n$,

$$\|PW\|_{L_\infty(\mathbb{R})}\leqslant C\left\{\frac{n}{a_n}T(a_n)^{1/2}\right\}^{1/p}\|PW\|_{L_p(\mathbb{R})}. \tag{3.10}$$

Proof. See Theorem 1.4 in [11]. ∎

LEMMA 3.4 (infinite–finite range inequality). *Let* $\beta\in\mathbb{R}$, *let* $\eta>0$, $s>1$, *and* $p>0$. *Then there exists* n_0 *such that for* $n\geqslant n_0$, $P\in\mathscr{P}_n$, *and*

$$g(x):=|PW|(x)\,\varphi_n(x/a_{\eta n})^\beta,\qquad x\in\mathbb{R}, \tag{3.11}$$

(a) $$\|\psi(g^p)\|_{L_\infty(\mathbb{R})}=\|\psi(g^p)\|_{L_\infty[-a_{sn},a_{sn}]}; \tag{3.12}$$

(b) $$\int_{-\infty}^{\infty}\psi(g^p(x))\,dx\leqslant(1+a_n^{-1})\int_{-a_{sn}}^{a_{sn}}\psi(g^p(x))\,dx. \tag{3.13}$$

Proof. (a) In view of the continuity of ψ and the compactness of $\{g(x): x\in\mathbb{R}\}$, we note that the sup's in (3.12) are attained. Furthermore, we see that (3.12) is equivalent to

$$\psi(\|g\|^p_{L_\infty(\mathbb{R})})=\psi(\|g\|^p_{L_\infty[-a_{sn},a_{sn}]}),$$

which in turn is equivalent to

$$\|g\|_{L_\infty(\mathbb{R})}=\|g\|_{L_\infty[-a_{sn},a_{sn}]}. \tag{3.14}$$

To prove (3.14), we note first that

$$\|g\|_{L_\infty[-a_{sn},a_{sn}]}\geqslant C_1T(a_n)^{-|\beta|}\|PW\|_{L_\infty[-a_{sn},a_{sn}]}, \tag{3.15}$$

this being a consequence of the fact that

$$T(a_n)^{-1}\leqslant\varphi_n(x/a_{\eta n})\leqslant C_2,\qquad |x|\leqslant a_{sn}. \tag{3.16}$$

Next, choose $\delta>0$ such that $1+\delta<s$. Let $\langle u\rangle$ denote the greatest integer $\leqslant u$. Then as

$$P(x)\,x^{\langle\delta n\rangle}\in\mathscr{P}_{n+\langle\delta n\rangle},$$

the Mhaskar–Saff identity (1.2) ensures that for $|x|\geqslant a_{sn}$,

$$|P(x)\,x^{\langle\delta n\rangle}W(x)|\leqslant\max\{|P(t)\,t^{\langle\delta n\rangle}W(t)|:|t|\leqslant a_{n+\langle\delta n\rangle}\},$$

so that

$$\begin{aligned} g(x) &= |PW|(x)\, \varphi_n(x/a_{\eta n})^{\beta} \\ &\leqslant \left(\frac{a_{n+\langle \delta n\rangle}}{|x|}\right)^{\langle \delta n\rangle} \|PW\|_{L_\infty[-a_n, a_n]}\, \varphi_n(x/a_{\eta n})^{\beta} \\ &\leqslant C_3 \left(\frac{a_{n+\langle \delta n\rangle}}{|x|}\right)^{\langle \delta n\rangle} \|g\|_{L_\infty[-a_{sn}, a_{sn}]}\, T(a_n)^{|\beta|} \left(\frac{|x|}{a_n}\right)^{2|\beta|} \\ &\qquad \text{(by (3.15))} \\ &\leqslant C_4 \left(\frac{a_{n(1+\delta)}}{|x|}\right)^{\langle \delta n\rangle - 2|\beta|} \|g\|_{L_\infty[-a_{sn}, a_{sn}]}\, T(a_n)^{|\beta|}. \end{aligned}$$

Now in view of (3.4),

$$\begin{aligned} &\left(\frac{a_{n(1+\delta)}}{|x|}\right)^{\langle \delta n\rangle/2 - 2|\beta|} T(a_n)^{|\beta|} \\ &\quad \leqslant \left(\frac{a_{n(1+\delta)}}{a_{sn}}\right)^{\langle \delta n\rangle/2 - 2|\beta|} T(a_n)^{|\beta|} \\ &\quad \leqslant (1 - C_5/T(a_n))^{C_6 n}\, T(a_n)^{|\beta|} \\ &\quad \leqslant \exp(-C_7 n/T(a_n) + |\beta| \log T(a_n)) \to 0, \end{aligned}$$

as $n \to \infty$, in view of (3.1). Thus,

$$g(x) \leqslant \left(\frac{a_{n(1+\delta)}}{|x|}\right)^{\langle \delta n\rangle/2} \|g\|_{L_\infty[-a_{sn}, a_{sn}]}, \tag{3.17}$$

for $|x| \geqslant a_{sn}$ and $n \geqslant n_0$. Then (3.14) follows.

(b) Let $1+\delta < s' < s$. We apply (3.17) to $t^L P(t)$, where L is a fixed positive integer chosen so that $Lp \geqslant 4$, and with s' replacing s. Then (3.17) yields, for $|x| > a_{s'(n+L)}$,

$$\begin{aligned} &|x|^L\, |PW|(x)\, \varphi_{n+L}(x/a_{\eta(n+L)})^{\beta} \\ &\quad \leqslant \left(\frac{a_{(n+L)(1+\delta)}}{|x|}\right)^{\langle \delta(n+L)\rangle/2} \\ &\qquad \times \max\left\{|t^L(PW)(t)|\, \varphi_{n+L}(t/a_{\eta(n+L)})^{\beta} : |t| \leqslant a_{s'(n+L)}\right\}. \end{aligned}$$

Using (3.6) and the fact that

$$a_{s'(n+L)} \leqslant a_{sn}, \qquad n \geqslant n_0,$$

we obtain, for $|x| \geqslant a_{sn}$,

$$g(x) \leqslant C_8 \left(\frac{a_{(n+L)(1+\delta)}}{|x|}\right)^{\langle \delta(n+L) \rangle/2} |x|^{-L} T(a_n)^{|\beta|}$$

$$\times \max\{|t^L (PW)(t)| : |t| \leqslant a_{n+L}\}$$

$$\leqslant C_9 \left(\frac{a_{(n+L)(1+\delta)}}{|x|}\right)^{\langle \delta(n+L) \rangle/2} |x|^{-L} T(a_n)^{|\beta|} \left\{\frac{n}{a_n} T(a_n)^{1/2}\right\}^{1/p}$$

$$\times \| t^L (PW)(t) \|_{L_p[-a_{2n}, a_{2n}]}$$

(by Lemma 3.3 and (3.9) of Lemma 3.2)

$$\leqslant C_{10} \left(\frac{a_{(n+L)(1+\delta)}}{|x|}\right)^{\langle \delta(n+L) \rangle/2} n^{C_{11}}$$

$$\times \left[\int_{-a_{2n}}^{a_{2n}} \left|\left(\frac{t}{a_{2n}}\right)^L (PW)(t)\, \varphi_n \left(\frac{t}{a_{\eta n}}\right)^{\beta}\right|^p (1+(xt)^2)^{-2}\, dt\right]^{1/p},$$

for $n \geqslant n_0$, where we have used (3.1) to estimate a_{2n} and $T(a_n)$ and have used $Lp \geqslant 4$. Now

$$\int_{-a_{2n}}^{a_{2n}} (1+(xt)^2)^{-2}\, dt = \frac{1}{|x|} \int_{-a_{2n}|x|}^{a_{2n}|x|} (1+u^2)^{-2}\, du \sim \frac{1}{|x|}, \qquad (3.18)$$

as $|x| \geqslant a_{sn}$. Furthermore, exactly as before (3.17), we see that for any fixed $A \in \mathbb{R}$, and uniformly for $|x| \geqslant a_{sn}$,

$$T(a_n)^A \left(\frac{a_{(n+L)(1+\delta)}}{|x|}\right)^{\langle \delta(n+L) \rangle/2} n^{C_{11}} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence for $n \geqslant n_0$,

$$g(x)^p \leqslant \frac{\int_{-a_{2n}}^{a_{2n}} \left|\frac{t}{a_{2n}}\right|^{Lp} |g(t)|^p (1+(xt)^2)^{-2}\, dt}{\int_{-a_{2n}}^{a_{2n}} (1+(xt)^2)^{-2}\, dt}.$$

Applying Jensen's inequality, (see, for example, [14, p. 24]) yields for $|x| \geqslant a_{sn}$,

$$\psi(g(x)^p) \leqslant \frac{\int_{-a_{2n}}^{a_{2n}} \psi\left(\left|\frac{t}{a_{2n}}\right|^{Lp} |g(t)|^p\right) (1+(xt)^2)^{-2}\, dt}{\int_{-a_{2n}}^{a_{2n}} (1+(xt)^2)^{-2}\, dt}.$$

Now as ψ is convex, we have, for $u \in [0, \infty)$ and $0 \leqslant y \leqslant 1$,

$$\psi(uy) = \psi(uy + 0(1-y)) \leqslant \psi(u)\, y + \psi(0)(1-y) = \psi(u)\, y.$$

Applying this and (3.18) yields

$$\psi(g(x)^p) \leqslant C_{12} \int_{-a_{2n}}^{a_{2n}} \psi(g(t)^p) \left|\frac{t}{a_{2n}}\right|^{Lp} \frac{|x|}{(1+(xt)^2)^2}\, dt. \qquad (3.19)$$

Since

$$\int_{|x| \geqslant a_{sn}} \frac{|x|}{(1+(xt)^2)^2}\, dx = \frac{2}{t^2} \int_{a_{sn}|t|}^{\infty} \frac{u}{(1+u^2)^2}\, du$$

$$\leqslant \frac{2}{t^2} \int_0^{\infty} \frac{u}{(1+u^2)^2}\, du,$$

and since $Lp \geqslant 4$, we obtain, on integrating for $|x| \geqslant a_{sn}$,

$$\int_{|x| \geqslant a_{sn}} \psi(g(x)^p)\, dx \leqslant C_{12} a_n^{-2} \int_{-a_{2n}}^{a_{2n}} \psi(g(t)^p) \left(\frac{|t|}{a_{2n}}\right)^{Lp-2} dt.$$

Then (3.13) follows. ∎

LEMMA 3.5. *Let $\beta \in \mathbb{R}$ and $\rho, A > 0$. There exists $R_n \in \mathscr{P}_{n-1}$, $n \geqslant 1$, such that uniformly for $n \geqslant 1$ and $|x| \leqslant A a_{\rho n}$,*

$$R_n(x) \sim \varphi_n(x/a_{\rho n})^{\beta}. \qquad (3.20)$$

Proof. We remark that we can actually choose R_n to be of degree $O(n^{\varepsilon})$ for each $\varepsilon > 0$. Let

$$h_n(z) := ((1-z^2)^2 + T(a_n)^{-2})^{\beta/2},$$

with branches chosen so that $h_n(z)$ is positive for $z \in \mathbb{R}$. The branchpoints lie where

$$1 - z^2 = \pm i T(a_n)^{-1},$$

or equivalently,

$$z = \pm 1 \pm \frac{i}{2T(a_n)} + O(T(a_n)^{-2}).$$

In any event, we may assume that the plane is cut so that $h_n(z)$ is analytic in the strip

$$\mathscr{S}_n = \{z : |\operatorname{Im} z| \leqslant (4T(a_n))^{-1}\},$$

for $n \geqslant n_0$. Let Γ_n be the ellipse with foci at $\pm A$, and minor semi-axis (that is, intercept on the positive y-axis) equal to

$$m_n := \frac{1}{2}\left(\frac{\rho_n}{A} - \frac{A}{\rho_n}\right),$$

where

$$\rho_n := A(1 + T(a_n)^{-2}).$$

Then

$$m_n = T(a_n)^{-2} + O(T(a_n)^{-4}), \qquad n \to \infty.$$

In particular, for n large enough, $\mathscr{S}_n$ contains Γ_n, and for some $C > 0$,

$$\max_{t \in \Gamma_n} |h_n(t)^{\pm 1}| \leqslant CT(a_n)^{|\beta|}.$$

Further, if $T_n(z)$ denotes the usual Chebyshev polynomial of degree n on $[-1, 1]$, then

$$\min_{t \in \Gamma_n} |T_n(t/A)| \geqslant C(\rho_n/A)^n \geqslant \exp(C_2 n/T(a_n)^2)$$
$$\geqslant \exp(n^{1/2}),$$

n large enough, by (3.1). Now, let $L_n(z) \in \mathscr{P}_{n-1}$ be the Lagrange interpolation polynomial to $h_n(z)$ at the zeros of $T_n(z/A)$. By the usual Hermite error formula, we have, for $z \in [-A, A]$,

$$|L_n(z)/h_n(z) - 1| = \left|\frac{1}{2\pi i}\int_{\Gamma_n} \frac{h_n(t)}{t-z}\,\frac{T_n(z/A)}{T_n(t/A)}\,\frac{dt}{h_n(z)}\right|$$
$$\leqslant C_2 T(a_n)^{2|\beta|}\, e^{-n^{1/2}} / \min_{t \in \Gamma_n} |t - z|$$
$$\leqslant C_3 T(a_n)^{2|\beta|+2}\, e^{-n^{1/2}} \to 0,$$

$n \to \infty$, by (3.1). Letting

$$R_n(x) := L_n(x/a_{\rho n}),$$

we have, for $n \geqslant n_0$ and $|x| \leqslant Aa_{\rho n}$,

$$R_n(x) \sim h_n(x/a_{\rho n}) \sim \varphi_n(x/a_{\rho n})^{\beta}.$$

Since $\varphi_n \sim 1$, $n < n_0$, we can choose $R_n \equiv 1$ for $n < n_0$. ▮

We can now prove a special case of Theorem 2.4:

LEMMA 3.6. *Let* $\rho > 0$, $l \geqslant 1$, *and* $A > 0$. *Then for* $n \geqslant 1$, $P \in \mathscr{P}_{ln}$, *and* $|x| \leqslant Aa_{3ln}$,

$$|PW|(x)^p \, \varphi_n(x/a_n)^{1/2} \leqslant C \frac{n}{a_n} \int_{-a_{3ln}}^{a_{3ln}} |PW|^p \,(t)\, dt. \tag{3.21}$$

Proof. If $\lambda_n(W^2, x)$ denotes the nth Christoffel function for W^2,

$$\lambda_n(W^2, x) := \min_{P \in \mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2 \,(t)\, dt/P^2(x),$$

then Theorem 1.2 in [7, p. 258] shows that

$$\sup_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \left[\left| 1 - \left(\frac{x}{a_n}\right)^2 \right|^{1/2} + T(a_n)^{-1/2} \right] \leqslant C_1 \frac{n}{a_n}.$$

Since uniformly for $x \in \mathbb{R}$ and $n \geqslant 1$

$$\varphi_n \left(\frac{x}{a_n}\right)^{1/2} \sim \left[\left| 1 - \left(\frac{x}{a_n}\right)^2 \right|^{1/2} + T(a_n)^{-1/2} \right],$$

we obtain

$$\sup_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x)\, \varphi_n \left(\frac{x}{a_n}\right)^{1/2} \leqslant C_1 \frac{n}{a_n}. \tag{3.22}$$

Then the definition of the Christoffel function ensures that, for each $P \in \mathscr{P}_{n-1}$ and $x \in \mathbb{R}$,

$$(PW)^2 \,(x)\, \varphi_n(x/a_n)^{1/2} \leqslant C_1 \frac{n}{a_n} \int_{-\infty}^{\infty} (PW)^2 \,(t)\, dt. \tag{3.23}$$

Now let us choose a positive integer k such that $2k \geqslant p$. Note that $W^k \in SE^*(3)$ and that $a_{nk}(W^k)$ (the nkth Mhaskar–Rahmanov–Saff number for W^k) equals $a_n(W)$. This is a direct consequence of (1.1).

Let $P \in \mathscr{P}_{2ln-1}$. Applying (3.23) to W^k and $P^k \in \mathscr{P}_{2lnk-1}$ yields, for $x \in \mathbb{R}$,

$$(PW)^{2k}(x)\,\varphi_{2ln}\left(\frac{x}{a_{2ln}}\right)^{1/2} \leqslant C\,\frac{2lnk}{a_{2ln}} \int_{-\infty}^{\infty} (PW)^{2k}(t)\,dt,$$

and by (3.6) and (3.9),

$$(PW)^{2k}(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{1/2} \leqslant C\,\frac{n}{a_n} \int_{-a_{3ln}}^{a_{3ln}} (PW)^{2k}(t)\,dt, \tag{3.24}$$

for $P \in \mathscr{P}_{2ln}$ and $x \in \mathbb{R}$. Next, by Lemma 3.5, we can find $R_n \in \mathscr{P}_{n-1}$, $n \geqslant 1$, such that

$$R_n(x) \sim \varphi_n\left(\frac{x}{a_n}\right)^{(2k-p)/(4kp)}, \qquad |x| \leqslant Aa_{3ln}.$$

Applying (3.24) to $PR_n \in \mathscr{P}_{2ln-1}$, where $P \in \mathscr{P}_{ln}$, yields, for $|x| \leqslant Aa_{3ln}$,

$$(PW)^{2k}(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{(2k-p)/(2p)+1/2}$$
$$\leqslant C_1\,\frac{n}{a_n} \int_{-a_{3ln}}^{a_{3ln}} (PW)^{2k}(t)\,\varphi_n\left(\frac{t}{a_n}\right)^{(2k-p)/(2p)} dt.$$

Then

$$\max\left\{|PW|(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{1/(2p)} : |x| \leqslant Aa_{3ln}\right\}^{2k}$$
$$\leqslant C_2\,\frac{n}{a_n} \int_{-a_{3ln}}^{a_{3ln}} |PW|^p(t) \left\{|PW|(t)\,\varphi_n\left(\frac{t}{a_n}\right)^{1/(2p)}\right\}^{2k-p} dt$$
$$\leqslant C_2\,\frac{n}{a_n} \int_{-a_{3ln}}^{a_{3ln}} |PW|^p(t)\,dt \max\left\{|PW|(t)\,\varphi_n\left(\frac{t}{a_n}\right)^{1/(2p)} : |t| \leqslant Aa_{3ln}\right\}^{2k-p}.$$

Hence (3.21). ▮

Recall that if

$$v(x) := (1-x^2)^{-1/2}, \qquad x \in (-1, 1),$$

is the Chebyshev weight, then $p_0(v, x) := \pi^{-1/2}$, and

$$p_n(v, x) := \left(\frac{2}{\pi}\right)^{1/2} T_n(x), \qquad n \geqslant 1,$$

are the associated orthonormal polynomials. The *nth kernel function* is

$$K_n(v, x, t) := \sum_{j=0}^{n-1} p_j(v, x)\, p_j(v, t),$$

and it satisfies

$$K_n(v, x, x) \sim n,\; n \geqslant 1,\; x \in [-1, 1], \tag{3.25}$$

$$|K_n(v, x, t)| \leqslant Cn,\; n \geqslant 1,\; x, t \in [-1, 1], \tag{3.26}$$

$$\int_{-1}^{1} K_n^2(v, x, t)\, v(t)\, dt = K_n(v, x, x), \tag{3.27}$$

and

$$\int_{-1}^{1} K_n^2(v, x, t)\, dt \sim n(|1 - x^2|^{1/2} + n^{-1}), \qquad n \geqslant 1,\; x \in [-1, 1]. \tag{3.28}$$

For (3.25) and (3.26), see [12, p. 108]. Of course (3.27) is a direct consequence of the orthonormality relations. For (3.28), see Theorem 2.2 in [5]. Using Lemma 3.6, we can now prove:

LEMMA 3.7. *Let $p > 0$, $l \geqslant 1$, $A \geqslant 1$, and $0 < s < A$. Let L be the least integer $\geqslant 2/p$, and let*

$$\rho := 3(l + L). \tag{3.29}$$

(a) *Then for $n \geqslant 1$, $P \in \mathscr{P}_{ln}$, and $|x| \leqslant Aa_{\rho n}$,*

$$|PW|^p(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{1/2} \leqslant \frac{C}{n\, a_n} \int_{-a_{\rho n}}^{a_{\rho n}} |PW|^p(t)\, K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt. \tag{3.30}$$

(b) *For $n \geqslant 1$, $P \in \mathscr{P}_{ln}$, and $|x| \leqslant sa_{\rho n}$,*

$$|PW|^p(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{1/2} \leqslant \frac{\int_{-Aa_{\rho n}}^{Aa_{\rho n}} (C_1 |PW|(t))^p\, K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt}{\int_{-Aa_{\rho n}}^{Aa_{\rho n}} K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt}. \tag{3.31}$$

Proof. (a) We apply Lemma 3.6 to

$$P(t)\, K_n^L\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) \in \mathscr{P}_{ln + Ln},$$

for fixed $|x| \leqslant Aa_{\rho n}$. For $|x| \leqslant Aa_{\rho n}$, Lemma 3.6 yields

$$|PW|^p(x)\, K_n^{Lp}\left(v, \frac{x}{Aa_{\rho n}}, \frac{x}{Aa_{\rho n}}\right) \varphi_n\left(\frac{x}{a_n}\right)^{1/2}$$

$$\leqslant C_1 \frac{n}{a_n} \int_{-a_{\rho n}}^{a_{\rho n}} |PW|^p(t)\, K_n^{Lp}\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt$$

$$\leqslant C_2 \frac{n}{a_n} n^{Lp-2} \int_{-a_{\rho n}}^{a_{\rho n}} |PW|^p(t)\, K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt,$$

by (3.26). Dividing by $K_n^{Lp}(v, x/Aa_{\rho n}, x/Aa_{\rho n})$ and using (3.25) yields (3.30).

(b) Now

$$\int_{-Aa_{\rho n}}^{Aa_{\rho n}} K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt = Aa_{\rho n} \int_{-1}^{1} K_n^2\left(v, \frac{x}{Aa_{\rho n}}, u\right) du \sim a_n n, \qquad (3.32)$$

by (3.28) for $|x| \leqslant sa_{\rho n}$, which implies $|x/(Aa_{\rho n})| \leqslant s/A < 1$. Then (3.30) yields (3.31). ∎

Proof of Theorem 2.4. Applying Jensen's inequality to (3.31) (and using (3.6)) yields, for $|x| \leqslant sa_{\rho n}$,

$$\psi\left(|PW|^p(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right)$$

$$\leqslant \frac{\int_{-Aa_{\rho n}}^{Aa_{\rho n}} \psi[(C_1 |PW|(t))^p]\, K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt}{\int_{-Aa_{\rho n}}^{Aa_{\rho n}} K_n^2\left(v, \frac{x}{Aa_{\rho n}}, \frac{t}{Aa_{\rho n}}\right) dt}$$

$$\leqslant C_2 \frac{n}{a_n} \int_{-Aa_{\rho n}}^{Aa_{\rho n}} \psi[(C_1 |PW|(t))^p]\, dt =: J,$$

by (3.26) and (3.32). We may choose $A > 1$ and $s = 1$. Then, we have, as $\rho \geqslant 3l$,

$$\max_{|x| \leqslant a_{3ln}} \psi\left[|PW|^p(x)\, \varphi_n\left(\frac{x}{a_{\beta_n}}\right)^{1/2}\right] \leqslant J.$$

By Lemma 3.4(a), we have

$$\max_{x \in \mathbb{R}} \psi\left[|PW|^p(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right] \leqslant J. \quad ∎$$

4. Proof of Theorems 2.2 and 2.3

Lemma 4.1. *Let $\alpha \geqslant \frac{1}{2}$. Then there exist $C > 0$ and n_0 such that for $n \geqslant n_0$ and $P \in \mathscr{P}_n$,*

$$\max_{x \in \mathbb{R}} |P'W|(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{\alpha} \leqslant C \frac{n}{a_n} \max_{x \in \mathbb{R}} |PW|(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{\alpha - 1/2}. \tag{4.1}$$

In particular,

$$\|P'W\|_{L_\infty(\mathbb{R})} \leqslant C_1 \frac{n}{a_n} T(a_n)^{1/2} \|PW\|_{L_\infty(\mathbb{R})}. \tag{4.2}$$

Proof. First, (4.1) is Theorem 1.5 in [11]. Then (4.2) (which is Theorem 1.3 in [4, p. 191]) follows. ∎

Lemma 4.2. *Let $p > 0$, $l \geqslant 1$, and let L be the least even integer $\geqslant 2/p$ and ρ be given by (3.29). Let $0 < s < 1$. Then for $n \geqslant n_0$, $P \in \mathscr{P}_{ln}$, and $|x| \leqslant a_{s\rho n}$,*

$$\begin{aligned} &\left\{ |P'W|(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{1/2} \right\}^p \varphi_n\left(\frac{x}{a_n}\right)^{1/2} \\ &\qquad \leqslant C_1 (na_n)^{-1} \int_{-a_{\rho n}}^{a_{\rho n}} \left(\frac{n}{a_n} |PW|(t)\right)^p K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) dt. \end{aligned} \tag{4.3}$$

Proof. By Lemma 4.1 with $\alpha = (1/2) + (1/2p)$, for $P \in \mathscr{P}_n$ and $x \in \mathbb{R}$,

$$\begin{aligned} \left\{ |P'W|(x)\, \varphi_n\left(\frac{x}{a_n}\right)^{1/2} \right\}^p \varphi_n\left(\frac{x}{a_n}\right)^{1/2} &\leqslant C \left(\frac{n}{a_n}\right)^p \max_{t \in \mathbb{R}} |PW|^p(t)\, \varphi_n\left(\frac{t}{a_n}\right)^{1/2} \\ &\leqslant C_1 \left(\frac{n}{a_n}\right)^{p+1} \int_{-\infty}^{\infty} |PW|^p(t)\, dt, \end{aligned} \tag{4.4}$$

by Theorem 2.4. Now we apply this to

$$P(t)\, K_n^L\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) \in \mathscr{P}_{ln + Ln},$$

where $P \in \mathscr{P}_{ln}$, and $|x| \leqslant a_{\rho n}$ is fixed. Let us set

$$K_n'(v, x, t) := \sum_{j=0}^{n-1} p_j(v, x)\, p_j'(v, t).$$

Then (4.4) yields

$$\left|P'(x)\,K_n^L\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)+P(x)\,LK_n^{L-1}\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)K_n'\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)\Big/a_{\rho n}\right|^p$$
$$\times\left\{W(x)\,\varphi_{(l+L)n}\left(\frac{x}{a_{(l+L)n}}\right)^{1/2}\right\}^p\varphi_{(l+L)n}\left(\frac{x}{a_{(l+L)n}}\right)^{1/2}$$
$$\leqslant C_2\left(\frac{(l+L)\,n}{a_{(l+L)n}}\right)^{p+1}\int_{-\infty}^{\infty}|PW|^p(t)\,K_n^{Lp}\left(v,\frac{x}{a_{\rho n}},\frac{t}{a_{\rho n}}\right)dt.$$

Dividing by $K_n^{Lp}(v, x/a_{\rho n}, x/a_{\rho n})\sim n^{Lp}$ and using (3.3), (3.6), and (3.9), yields for $P\in\mathscr{P}_{ln}$ and $|x|\leqslant a_{\rho n}$,

$$\left|P'(x)+P(x)\,LK_n^{-1}\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)K_n'\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)\Big/a_{\rho n}\right|^p$$
$$\times\left\{W(x)\,\varphi_n\left(\frac{x}{a_{\rho n}}\right)^{1/2}\right\}^p\varphi_n\left(\frac{x}{a_{\rho n}}\right)^{1/2}$$
$$\leqslant C_3\left(\frac{n}{a_n}\right)^{p+1}n^{-Lp}\int_{-a_{\rho n}}^{a_{\rho n}}|PW|^p(t)\,K_n^{Lp}\left(v,\frac{x}{a_{\rho n}},\frac{t}{a_{\rho n}}\right)dt$$
$$\leqslant C_4\left(\frac{n}{a_n}\right)^{p+1}n^{-2}\int_{-a_{\rho n}}^{a_{\rho n}}|PW|^p(t)\,K_n^2\left(v,\frac{x}{a_{\rho n}},\frac{t}{a_{\rho n}}\right)dt, \tag{4.5}$$

by (3.26) and as $Lp\geqslant 2$. Next, we note that by Bernstein's classical inequality [1], for $|x/a_{\rho n}|<1$,

$$\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{1/2}\left|K_n'\left(v,\frac{x}{a_{\rho n}},\frac{x}{a_{\rho n}}\right)\right|\leqslant n\max_{|t|\leqslant a_{\rho n}}\left|K_n\left(v,\frac{x}{a_{\rho n}},\frac{t}{a_{\rho n}}\right)\right|\leqslant C_5 n^2.$$

Then (4.5) and (3.25) yield, for $|x|<a_{\rho n}$,

$$\left\{|P'W|(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{1/2}\right\}^p\varphi_n\left(\frac{x}{a_n}\right)^{1/2}$$
$$\leqslant C_6(na_n)^{-1}\int_{-a_{\rho n}}^{a_{\rho n}}\left(\frac{n}{a_{\rho n}}|PW|(t)\right)^p K_n^2\left(v,\frac{x}{a_{\rho n}},\frac{t}{a_{\rho n}}\right)dt$$
$$+C_6\left\{|PW|(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{1/2}\right\}^p\varphi_n\left(\frac{x}{a_n}\right)^{1/2}\left(\frac{n}{a_n}\right)^p\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{-p/2}$$
$$=:\tau_1+\tau_2, \tag{4.6}$$

say. Next, note that for $|x| < a_{s\rho n}$,

$$\varphi_n\left(\frac{x}{a_n}\right)\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{-1} \sim \varphi_n\left(\frac{x}{a_{\rho n}}\right)\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{-1} \qquad \text{(by (3.6))}$$

$$= 1+\left[T(a_n)\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|\right]^{-1}.$$

Since

$$\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right| \geqslant \left|1-\left(\frac{a_{s\rho n}}{a_{\rho n}}\right)^2\right| \geqslant C_7/T(a_n),$$

we obtain

$$\tau_2 \leqslant C_8 |PW|^p(x)\,\varphi_n\left(\frac{x}{a_n}\right)^{1/2}\left(\frac{n}{a_n}\right)^p$$

$$\leqslant C_9(na_n)^{-1}\int_{-a_{\rho n}}^{a_{\rho n}}\left(\frac{n}{a_{\rho n}}|PW|(t)\right)^p K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right)dt,$$

by (3.30) of Lemma 3.7, with $A=1$. Together with (4.6), this yields (4.3). ∎

Proof of Theorem 2.3. Now let $l=1$ and ρ be given by (3.29), and let $0<s<1$. Then for $n \geqslant n_0$ and $|x| \leqslant a_{s\rho n}$,

$$\int_{-a_{\rho n}}^{a_{\rho n}} K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right)dt$$

$$= a_{\rho n}\int_{-1}^{1} K_n^2\left(v, \frac{x}{a_{\rho n}}, u\right)du$$

$$\sim na_n\left(\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{1/2} + n^{-1}\right) \qquad \text{(by (3.28))}$$

$$\sim na_n\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{1/2} \sim na_n\varphi_n\left(\frac{x}{a_{\rho n}}\right)^{1/2} \sim na_n\varphi_n\left(\frac{x}{a_n}\right)^{1/2},$$

since $|1-(x/a_{\rho n})^2| \geqslant C/T(a_n) > n^{-2}$, and by (3.6). Then we deduce from (4.3) that

$$\left\{|P'W|(x)\,\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right\}^p$$

$$\leqslant \frac{\int_{-a_{\rho n}}^{a_{\rho n}}\left(C_2\frac{n}{a_n}|PW|(t)\right)^p K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right)dt}{\int_{-a_{\rho n}}^{a_{\rho n}} K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right)dt},$$

for $n \geqslant n_0$, $P \in \mathscr{P}_n$, and $|x| \leqslant a_{s\rho n}$. Applying Jensen's inequality yields

$$\psi\left(\left\{|P'W|(x)\,\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right\}^p\right)$$

$$\leqslant \frac{\int_{-a_{\rho n}}^{a_{\rho n}} \psi\left[\left(C_2 \frac{n}{a_n} |PW|(t)\right)^p\right] K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) dt}{\int_{-a_{\rho n}}^{a_{\rho n}} K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) dt}$$

$$\leqslant C_3\left(na_n\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{1/2}\right)^{-1} \int_{-a_{\rho n}}^{a_{\rho n}} \psi\left(\left\{C_2 \frac{n}{a_n} |PW|(t)\right\}^p\right)$$

$$\times K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) dt.$$

Integrating for x from $-a_{s\rho n}$ to $a_{s\rho n}$, and using

$$\int_{-a_{\rho n}}^{a_{\rho n}} K_n^2\left(v, \frac{x}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right)\left|1-\left(\frac{x}{a_{\rho n}}\right)^2\right|^{-1/2} dx$$

$$= a_{\rho n} \int_{-1}^{1} K_n^2\left(v, u, \frac{t}{a_{\rho n}}\right) |1-u^2|^{-1/2}\, du$$

$$= a_{\rho n} K_n\left(v, \frac{t}{a_{\rho n}}, \frac{t}{a_{\rho n}}\right) \leqslant C_4 n a_n$$

(see (3.27) and (3.25)), we obtain

$$\int_{-a_{s\rho n}}^{a_{s\rho n}} \psi\left(\left\{|P'W|(x)\,\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2}\right\}^p\right) dx$$

$$\leqslant C_5 \int_{-a_{\rho n}}^{a_{\rho n}} \psi\left(\left\{C_2 \frac{n}{a_n} |PW|(t)\right\}^p\right) dt,$$

for all $P \in \mathscr{P}_n$. Here we may choose $s \in (0, 1)$ so that

$$s\rho n = s3(1+L)\,n = Sn,$$

with $S > 1$. Then (3.13) yields (2.13). ∎

Proof of Theorem 2.2. First, (2.8) is the special case $\psi(t) = t$ of (2.13). Since

$$\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2} \geqslant C_1/T(a_n)^{1/2},$$

and

$$Q'(a_n) \sim \frac{n}{a_n} T(a_n)^{1/2}$$

(see (3.5)), we can then deduce (2.9). Since

$$\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{1/2} \geqslant \left|1-\left(\frac{x}{a_{\beta n}}\right)^2\right|^{1/2},$$

(2.10) also follows. ∎

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